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Picard and Adomian solutions of nonlinear fractional differential equations system containing Atangana – Baleanu derivative

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Abstract

In this paper, we apply two methods for solving nonlinear system of fractional differential equations (FDEs); these two methods are Picard and Adomian decomposition methods (ADM). The type of fractional derivative in this system will be the Atangana–Baleanu derivative. The existence and uniqueness of the solution will be proved. In addition, the convergence of ADM series solution and the maximum expected error will be discussed. Some numerical examples will be solved by using these two method and a comparison between their solutions will be done. There exist an important application to these types of systems, this application is the fractional-order rabies model and it will be solved here. From the obtained results, it is noticed that the obtained results from using these two methods are coincide with each other, and also these results are coincide with the obtained results from the classical fractional derivatives such as Caputo sense.

Keywords: Fractional differential equations, Atangana – Baleanu derivative, Picard and Adomian decomposition methods, Existence and uniqueness, Error analysis, Fractional-order rabies model

Introduction

Fractional Differential equations have many applications in engineering and science; some of them are fluid flow [1, 2], electrical networks, control theory [3, 4], electromagnetic theory, viscoelasticity [5, 6], fractals theory, potential theory [2, 7], biology, chemistry [8, 9], optical and neural network systems [10–12]. In this paper, Picard [13–15] and Adomian decomposition methods [16–20] will be used to solve these type of systems. These two methods have many advantages; they efficiently work with different types of linear and nonlinear equations [21–24] in deterministic or stochastic [25–27] fields and gives an analytic solution for all these types of equations without linearization or discretization [28–30].

The paper will be organized as follows:

In Methods section, Picard and ADM will be introduced as the two used methods to solve the system under consideration. In Results and discussion section, Existence and uniqueness of the solution will be proved, convergence of ADM series solution and error analysis will be discussed. Finally, an important application to these types of systems will be solved which is fractional-order rabies model and other numerical examples will be solved by using these two methods and a comparison between their results will be illustrated.

Methods

In this research, two methods will be used to solve a nonlinear system of fractional differential equations containing Atangana–Baleanu derivative. The first method is ADM and the second method is Picard method.

Formulation of the problem

Consider a system of nonlinear FDEs of the form,

$$ {}^{AB}\mathcal{D}_t^{\alpha} y_i(t) + g_i(t) f_i\left(\bar{y}(t)\right) = x_i(t), \quad \alpha \in (n-1, n),\ i = 1, 2, \ldots, n. \tag{1} $$

Subject to the initial conditions,

$$ {y_i}^{(j-1)}(0) = c_j, \quad j = 1, 2, \ldots, n. \tag{2} $$

Where $\bar{y} = \left(y_1, y_2, \ldots, y_n\right)$ and ${}^{AB}\mathcal{D}_t^{\alpha}(.)$ is fractional derivative of Atangana–Baleanu sense that defined as:

$$ {}^{AB}\mathcal{D}_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_{\alpha} \left(\frac{-\alpha(t-s)}{1-\alpha} \right)^{\alpha} f'(s) ds $$

Where $B(\alpha) > 0$, is a normalization function satisfying $B(0) = B(1) = 1$ and E_α is the Mittag–Leffler function of one variable. The corresponding fractional integral defined by see [3, 4],

$$ {}^{AB}I^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t f(s)(t-s)^{\alpha-1} ds, \quad 0 < \alpha < 1. $$

And

$$ \left({}^{AB}I^{\alpha}\right)\left({}^{AB}\mathcal{D}_t^{\alpha}\right) f(t) = f(t) - f(a) $$

Now applying the integrating operator of order α to the system (1)-(2), this reduces it to the system of fractional integral equations,

$$ \begin{aligned} y_i(t) = \sum_{i=1}^{n} \tfrac{c_i}{\Gamma(\alpha)} t^{\alpha-1} + \tfrac{1-\alpha}{B(\alpha)} x_i(t) + \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x_i(\tau) d\tau \\ - \tfrac{1-\alpha}{B(\alpha)} g_i(t) f_i\left(\bar{y}(t)\right) - \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g_i(\tau) f_i\left(\bar{y}(\tau)\right) d\tau \end{aligned} \tag{3} $$

Assume that $x_i(t)$ bounded $\forall t \in I = [0, T]$, $T \in R^+$, $\left|g_i(\tau)\right| \le M_i \forall 0 \le \tau \le T$, M_i are finite constants and $f_i(\bar{y})$ satisfy Lipschitz condition with Lipschitz constants L_i such as,

$$ \left|f_i(\bar{y}) - f_i(\bar{z})\right| \le L_i \sum_{k=1}^{n} \left|y_k - z_k\right| \tag{4} $$

The first method: ADM

Applying ADM depends on replacing the nonlinear term with its corresponding Adomian polynomials as follows,

$$f_i(\overline{y}) = \sum_{k=0}^{\infty} A_{ik}(y_{i0}, y_{i1}, \ldots, y_{ik}) \tag{5}$$

Where,

$$A_{ik}(y_{i0}, y_{i1}, \ldots, y_{ik}) = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[f_i\left(\sum_{j=0}^{\infty}\lambda^j y_{ij}\right)\right]_{\lambda=0} \tag{6}$$

Substitute from Eq. (5) into Eq. (3), we get

$$\begin{aligned} y_i(t) = &\sum_{i=1}^{n}\tfrac{c_i}{\Gamma(\alpha)}t^{\alpha-1} + \tfrac{1-\alpha}{B(\alpha)}x_i(t) + \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x_i(\tau)d\tau \\ &-\tfrac{1-\alpha}{B(\alpha)}g_i(t)\sum_{k=0}^{\infty}A_{ik} - \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g_i(\tau)\sum_{k=0}^{\infty}A_{ik}\,d\tau \end{aligned} \tag{7}$$

Let $y_i(t) = \sum_{k=0}^{\infty} y_{ik}(t)$ in (7) we get,

$$y_{i0}(t) = \sum_{i=1}^{n}\frac{c_i}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1-\alpha}{B(\alpha)}x_i(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x_i(\tau)d\tau, \tag{8}$$

$$\begin{aligned} &y_{ik}(t) = -\tfrac{1-\alpha}{B(\alpha)}g_i(t)A_{i(k-1)} \\ &-\tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g_i(\tau)A_{i(k-1)}\,d\tau, k \geq 1. \end{aligned} \tag{9}$$

Finally, the ADM series solution will be,

$$y_i(t) = \sum_{k=0}^{\infty} y_{ik}(t) \tag{10}$$

Existence and uniqueness theorem

Let $E = ((I), \mathbb{R}^{(n)})$ be the Banach space of continuous functions defined on the compact interval I that are valued in $\mathbb{R}^{(n)}$. On $\mathbb{R}^{(n)}$ is considered the norm $\|\boldsymbol{y}\| = \sum_{i=1}^{n}|y_i|$ where $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^{(n)}$. If $\mathbf{y} \in E$ and $\mathbf{y}(\mathrm{t}) = (y_1(t), y_2(t), \ldots, y_n(t))$ then $\|\boldsymbol{y}\| = \sum_{i=1}^{n}\max_{t\in J}|y_i(t)|$.

Theorem 1 *The system* (1) *and* (2) *has a unique solution whenever* $0 < \beta < 1$, $\beta = \frac{LM}{B(\alpha)}[(1-\alpha) + \frac{\alpha T^{\alpha}}{\Gamma(\alpha+1)}]$ *where* $L = \sum_{m=1}^{n} L_m, M = \max\{M_1, M_2, \ldots, M_n\}$.

Proof Equation (3) can written as,

$$\begin{aligned} \boldsymbol{y}(t) = &\sum_{i=1}^{n}\tfrac{c_i}{\Gamma(\alpha)}t^{\alpha-1} + \tfrac{1-\alpha}{B(\alpha)}x_i(t) + \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x_i(\tau)d\tau \\ &-\tfrac{1-\alpha}{B(\alpha)}g_i(t)f_i(\boldsymbol{y}(t)) - \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g_i(\tau)f_i(\boldsymbol{y}(\tau))d\tau \end{aligned}$$

Where,

$$\begin{aligned}
&\boldsymbol{x}(t) = (x_1, x_2, \ldots, x_n)', \\
&\boldsymbol{g}(t) = diag\{g_1, g_2, \ldots, g_n\}, \\
&\boldsymbol{f}(\boldsymbol{y}(t)) = \big(f_1(\boldsymbol{y}), f_2(\boldsymbol{y}), \ldots, f_n(\boldsymbol{y})\big)'.
\end{aligned}$$

The mapping $R : E \to E$ defined as,

$$\begin{aligned}
R\boldsymbol{y}(t) = &\sum_{i=1}^{n} \tfrac{c_i}{\Gamma(\alpha)} t^{\alpha-1} + \tfrac{1-\alpha}{B(\alpha)} x(t) + \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \boldsymbol{x}(\tau) d\tau \\
&- \tfrac{1-\alpha}{B(\alpha)} \boldsymbol{g}(t) \boldsymbol{f}(\boldsymbol{y}(t)) - \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \boldsymbol{g}(\tau) \boldsymbol{f}(\boldsymbol{y}(\tau)) d\tau
\end{aligned}$$

Let $Y, Z \in E$:

$$\begin{aligned}
&\|RY(t) - RZ(t)\| = \| - \tfrac{1-\alpha}{B(\alpha)} \boldsymbol{g}(t) \big(\boldsymbol{f}(\overline{y}) - \boldsymbol{f}(\overline{z})\big) \\
&- \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \boldsymbol{g}(\tau) \boldsymbol{f}(\boldsymbol{y}(\tau)) d\tau \| \\
&\leq \tfrac{1-\alpha}{B(\alpha)} \|\boldsymbol{g}(\tau)\| \|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{z})\| \\
&+ \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\boldsymbol{g}(\tau)\| \|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{z})\| d\tau \\
&\leq \tfrac{(1-\alpha)M}{B(\alpha)} \textstyle\sum_{m=1}^{n} L_m \left(\sum_{m=1}^{n} \max\limits_{t \in J} \left| f_m(\boldsymbol{y}) - f_m(\boldsymbol{z}) \right| \right) \\
&+ \tfrac{\alpha M}{B(\alpha)\Gamma(\alpha)} \textstyle\sum_{m=1}^{n} L_m \left(\sum_{m=1}^{n} \max\limits_{t \in J} \left| f_m(\boldsymbol{y}) - f_m(\boldsymbol{z}) \right| \right) \int_0^t (t-\tau)^{\alpha-1} d\tau \\
&\leq \tfrac{(1-\alpha)M}{B(\alpha)} \textstyle\sum_{m=1}^{n} L_m \left(\sum_{m=1}^{n} \max\limits_{t \in J} \left| y_k - z_k \right| \right) \\
&+ \tfrac{\alpha M T^\alpha}{B(\alpha)\Gamma(\alpha+1)} \textstyle\sum_{m=1}^{n} L_m \left(\sum_{m=1}^{n} \max\limits_{t \in J} \left| y_k - z_k \right| \right) \\
&\leq \left[\tfrac{(1-\alpha)ML}{B(\alpha)} + \tfrac{\alpha M T^\alpha L}{B(\alpha)\Gamma(\alpha+1)} \right] \|\boldsymbol{y} - \boldsymbol{z}\| \\
&\leq \tfrac{LM}{B(\alpha)} \left[(1-\alpha) + \tfrac{\alpha T^\alpha}{\Gamma(\alpha+1)} \right] \|\boldsymbol{y} - \boldsymbol{z}\| \\
&\leq \beta \|Y - Z\|
\end{aligned}$$

Under the condition, $0 < \beta < 1$, the mapping R is contraction and there exist a unique solution of the system (1)-(2).

Proof of convergence

Theorem 2 *The series solution* (10) *of the system* (1)-(2) *using ADM converges if* $\left| y_{i1} \right| < \infty$ *and* $0 < \beta < 1$, $\beta = \frac{LM}{B(\alpha)}[(1-\alpha) + \frac{\alpha T^\alpha}{\Gamma(\alpha+1)}]$, *where* $L = \sum_{k=1}^{n} L_k$, $M = \max\{M_1, M_2, \ldots, M_n\}$.

Proof Define a sequence $\{S_{ip}\}$ as, $S_{ip} = \sum_{k=0}^{p} y_{ik}(t)$ is the sequence of partial sums from the series solution $\sum_{k=0}^{\infty} y_{ik}(t)$, we have,

$$f(S_{ip}) = \sum_{k=0}^{p} A_{ik}(y_{i0}, y_{i1}, \ldots, y_{ip})$$

Let S_{ip} and S_{iq} be two arbitrary partial sums such that $p > q$. Now, we are going to prove that $\{S_{ip}\}$ is a Cauchy sequence in this Banach space.

$$\begin{aligned}
&\|S_{ip}-S_{iq}\| = \sum_{k=1}^{n}\max_{t\in J}\left|S_{kp}-S_{kq}\right| \\
&= \sum_{k=1}^{n}\max_{t\in J}\left|\sum_{j=q+1}^{p} y_{kj}(t)\right| \\
&= \sum_{k=1}^{n}\max_{t\in J}\left|\sum_{j=q+1}^{p}\left[\tfrac{1-\alpha}{B(\alpha)}g_k(t)A_{i(k-1)}+\tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}g_k(\tau)A_{k(j-1)}\,d\tau\right]\right| \\
&= \sum_{k=1}^{n}\max_{t\in J}\left|\textstyle\sum_{j=q+1}^{p}\tfrac{1-\alpha}{B(\alpha)}g_k(t)\displaystyle\sum_{j=q+1}^{p}A_{k(j-1)}+\tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t g_k(\tau)(t-\tau)^{\alpha-1}\textstyle\sum_{j=q+1}^{p}A_{k(j-1)}d\tau\right| \\
&= \sum_{k=1}^{n}\max_{t\in J}\left|\sum_{j=q+1}^{p}\tfrac{1-\alpha}{B(\alpha)}g_k(t)\sum_{j=q}^{p-1}A_{kj}+\tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t g_k(\tau)(t-\tau)^{\alpha-1}\sum_{j=q}^{p-1}A_{kj}d\tau\right| \\
&= \sum_{k=1}^{n}\max_{t\in J}\left|\sum_{j=q+1}^{p}\tfrac{1-\alpha}{B(\alpha)}g_k(t)\left[f\left(S_{k(p-1)}\right)-f\left(S_{k(q-1)}\right)\right]\right. \\
&\left.+\tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t g_k(\tau)(t-\tau)^{\alpha-1}[f(S_{k(p-1)})-f(S_{k(q-1)})]d\tau\right| \\
&\le \tfrac{1-\alpha}{B(\alpha)}\sum_{k=1}^{n}\max_{t\in J}[\left|g_k(t)\right|\left|f\left(S_{k(p-1)}\right)-f\left(S_{k(q-1)}\right)\right|] \\
&+\tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\textstyle\sum_{k=1}^{n}\max_{t\in J}[\int_0^t\left|g_k(t)\right|\left|(t-\tau)^{\alpha-1}\right|\left|f\left(S_{k(p-1)}\right)-f\left(S_{k(q-1)}\right)\right|d\tau] \\
&\le \tfrac{(1-\alpha)ML}{B(\alpha)}\max_{t\in J}\sum_{j=1}^{n}\left|S_{j(p-1)}-S_{j(q-1)}\right| \\
&+\tfrac{\alpha ML}{B(\alpha)\Gamma(\alpha)}\max_{t\in J}\textstyle\sum_{j=1}^{n}\left|S_{j(p-1)}-S_{j(q-1)}\right|\left|\int_0^t(t-\tau)^{\alpha-1}d\tau\right. \\
&\le \tfrac{LM}{B(\alpha)}\left[(1-\alpha)+\tfrac{\alpha T^{\alpha}}{\Gamma(\alpha+1)}\right]\|S_{i(p-1)}-S_{i(q-1)}\| \\
&\le \beta\|S_{i(p-1)}-S_{i(q-1)}\|
\end{aligned}$$

Let $p=q+1$ then,

$$\|S_{i(q+1)}-S_{iq}\|\le\beta\|S_{iq}-S_{i(q-1)}\|\le\beta^2\|S_{i(q-1)}-S_{i(q-2)}\|\le\cdots\le\beta^q\|S_{i1}-S_{i0}\|$$

Using the triangle inequality,

$$\begin{aligned}
\|S_{ip}-S_{iq}\| &\le \|S_{i(q+1)}-S_{iq}\|+\|S_{i(q+2)}-S_{i(q+1)}\|+\cdots+\|S_{ip}-S_{i(p-1)}\| \\
&\le \left[\beta^q+\beta^{q+1}+\cdots+\beta^{p-1}\right]\|S_{i1}-S_{i0}\| \\
&\le \beta^q\left[1+\beta+\cdots+\beta^{p-q-1}\right]\|S_{i1}-S_{i0}\| \\
&\le \beta^q\left[\tfrac{1-\beta^{p-q}}{1-\beta}\right]\|y_{i1}(t)\|
\end{aligned}$$

Since, $0<\beta<1$ and $p>q$ then, $(1-\beta^{p-q})\le 1$. Consequently,

$$\begin{aligned}
\|S_{ip}-S_{iq}\| &\le \tfrac{\beta^q}{1-\beta}\|y_{i1}(t)\| \\
&\le \tfrac{\beta^q}{1-\beta}\max_{t\in J}\left|y_{i1}(t)\right|
\end{aligned}$$

If $\left|y_{i1}(t)\right|<\infty$ and as $q\to\infty$ then, $\|S_{ip}-S_{iq}\|\to 0$ and hence, $\left\{S_{ip}\right\}$ is a Cauchy sequence in this Banach space so, the series $\sum_{k=0}^{\infty}y_{ik}(t)$ converges.

Error analysis

Theorem 3 *The maximum absolute truncation error of the series solution* (10) *to the system* (1)-(2) *estimated to be,*

$$\max_{t\in J}\left|y_i(t)-\sum_{k=0}^{q}y_{ik}(t)\right|\le\frac{\beta^q}{1-\beta}\max_{t\in J}\left|y_{i1}(t)\right|$$

Proof From Theorem 2 we get that

$$\|S_{ip} - S_{iq}\| \leq \frac{\beta^q}{1-\beta} \max_{t \in J} |y_{i1}(t)|$$

If $S_{ip} = \sum_{k=0}^{p} y_{ik}(t)$ as $p \to \infty$ then,$S_{ip} \to y_i(t)$ so,

$$\|y_i(t) - S_{iq}\| \leq \frac{\beta^q}{1-\beta} \max_{t \in J} |y_{i1}(t)|.$$

Hence the maximum absolute truncation error in the interval J is,

$$\max_{t \in J} \left| y_i(t) - \sum_{k=0}^{q} y_{ik}(t) \right| \leq \frac{\beta^q}{1-\beta} \max_{t \in J} |y_{i1}(t)|$$

The second method: Picard method

Applying Picard method to the system (3), the solution is constructed by the sequence,

$$y_{i0}(t) = \sum_{i=1}^{n} \frac{c_i}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1-\alpha}{B(\alpha)} x_i(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x_i(\tau) d\tau, \tag{11}$$

$$\begin{aligned} &y_{ik}(t) = y_{i0}(t) - \tfrac{1-\alpha}{B(\alpha)} g_i(t) f_i\left(y_{i(k-1)}(\tau)\right) \\ &-\tfrac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g_i(\tau) f_i\left(y_{i(k-1)}(\tau)\right) d\tau, k \geq 1. \end{aligned} \tag{12}$$

Finally, the Picard solution will be,

$$y_i(t) = \lim_{k \to \infty} y_{ik}(t) \tag{13}$$

Results and discussion

Example 1. Fractional-order rabies model

The fractional-order rabies model,

$$\begin{aligned} {}^{AB}\mathcal{D}_t^{\alpha} y_1 &= -by_1y_2, \\ {}^{AB}\mathcal{D}_t^{\alpha} y_2 &= by_1y_2 - dy_2, \end{aligned} \tag{14}$$

Subject to the initial conditions,

$$y_1(0) = 1, \; y_2(0) = 2,$$

Was discussed before in [22], it solved by using Adams-type predictor–corrector method. Now, we will solve it by using ADM.

1- **ADM solution:**

Using ADM to system (14) leads to the following solution algorithm,

$$\begin{aligned} &y_{1,0}=1, \quad y_{1,j+1}=-b^{AB}I^{\alpha}(A_{1,j}), \\ &y_{2,0}=2, \quad y_{2,j+1}={}^{AB}I^{\alpha}\left(bA_{1,j}-dy_{2,j}\right), \end{aligned} \tag{15}$$

Where $A_{1,j}$ represent the Adomian polynomials of the nonlinear term y_1y_2.

Moreover, the final solution will be,

$$y_1=\sum_{i=0}^{n}y_{1,i}, y_2=\sum_{i=0}^{n}y_{2,i}.$$

2- **Picard Solution:**

Using Picard method to the system (14), the solution algorithm will be,

$$\begin{aligned} &y_{1,0}=1, \quad y_{1,j+1}=y_{1,0}-b^{AB}I^{\alpha}[y_{1,j}y_{2,j}], \\ &y_{2,0}=2, \quad y_{2,j+1}=y_{2,0}+{}^{AB}I^{\alpha}\left[by_{1,j}y_{2,j}-dy_{2,j}\right]. \end{aligned} \tag{16}$$

Moreover, the final solution will be,

$$y_1=\lim_{n\to\infty}y_{1,n}, y_2=\lim_{n\to\infty}y_{2,n}.$$

Figure 1a and b show ADM and Picard solutions of y_1 and y_2 where ($n = 5, b = d = 1$) at ($\alpha = 0.8, 0.9$).

From these two figures, we see that ADM solutions of (y_1 *and* y_2) are coincide with Picard solutions at the same values of α.

Example 2. Consider the following nonlinear system of FDEs,

$$\begin{aligned} &{}^{AB}\mathcal{D}_t^{0.5}({}^{AB}\mathcal{D}_t^{0.5}y_1)=1+y_2^3-t^6, \\ &{}^{AB}\mathcal{D}_t^{0.5}({}^{AB}\mathcal{D}_t^{0.5}y_2)=y_1+t, \\ &{}^{AB}\mathcal{D}_t^{0.5}({}^{AB}\mathcal{D}_t^{0.5}y_3)=3y_1^2, \end{aligned} \tag{17}$$

Subject to the initial conditions,

$$y_k(0)=0, \quad k=1,2,3.$$

Which has the exact solution $y_1(t)=t, y_2(t)=t^2$ *and* $y_3(t)=t^3$.

1- **ADM Solution:**

Apply ${}^{AB}I^{\alpha}$ to the system (17), then using ADM and replace each nonlinear term by its corresponding Adomian polynomials we obtain,

$$\begin{aligned} &y_{1,0}=t-\tfrac{t^7}{7}, \quad y_{1,j+1}={}^{AB}I^1[A_{1,j}], \\ &y_{2,0}=\tfrac{t^2}{2}, \quad y_{2,j+1}={}^{AB}I^1[y_{1,j}], \\ &y_{3,0}=0, \quad y_{3,j+1}={}^{AB}I^1[3A_{2,j}]. \end{aligned} \tag{18}$$

Moreover, the final solution will be,

$$y_1=\sum_{i=0}^{\infty}y_{1,n}, y_2=\sum_{i=0}^{\infty}y_{2,n}, y_3=\sum_{i=0}^{\infty}y_{3,n}.$$

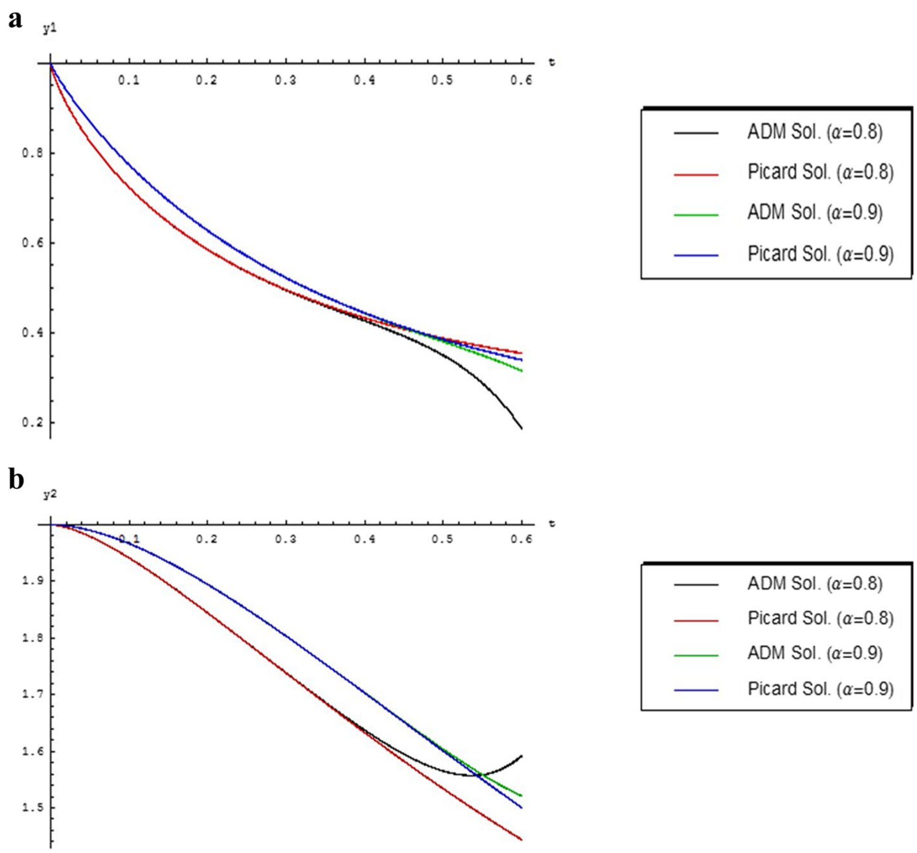

Fig. 1 **a** ADM and Picard solution of $y_1 (\alpha = 0.8, 0.9)$. **b** ADM and Picard solution of $y_2 (\alpha = 0.8, 0.9)$

2- **Picard Solution:**

Using Picard method to the system (17), the solution algorithm will be,

$$\begin{aligned} &y_{1,0} = t - \tfrac{t^7}{7}, \quad y_{1,j+1} = y_{1,0} + {}^{AB}I^1[(y_{2,j})^3], \\ &y_{2,0} = \tfrac{t^2}{2}, \quad y_{2,j+1} = y_{2,0} + {}^{AB}I^1\left[y_{1,j}\right], \\ &y_{3,0} = 0, \quad y_{3,j+1} = y_{3,0} + {}^{AB}I^1[3(y_{1,j})^2]. \end{aligned} \tag{19}$$

Moreover, the final solution will be,

$$y_1 = \lim_{n\to\infty} y_{1,n}, y_2 = \lim_{n\to\infty} y_{2,n}, y_3 = \lim_{n\to\infty} y_{3,n}.$$

Figure 2a-c show Picard and exact solutions of y_1, y_2 and y_3 ($n = 5$). While, Fig. 2d-f show ADM and exact solutions of y_1, y_2 and y_3 ($n = 5$).

Tables 1, 2 and 3 show the relative absolute error between exact solutions, Picard and ADM solutions of y_1, y_2 and y_3. A comparison between Picard with exact solutions and ADM with exact solutions are shown from these results that Picard method give more accurate results than ADM but ADM take less time than Picard (ADM time = 0.235, Picard time = 0.455).

Example 3. Consider the following nonlinear system of FDEs,

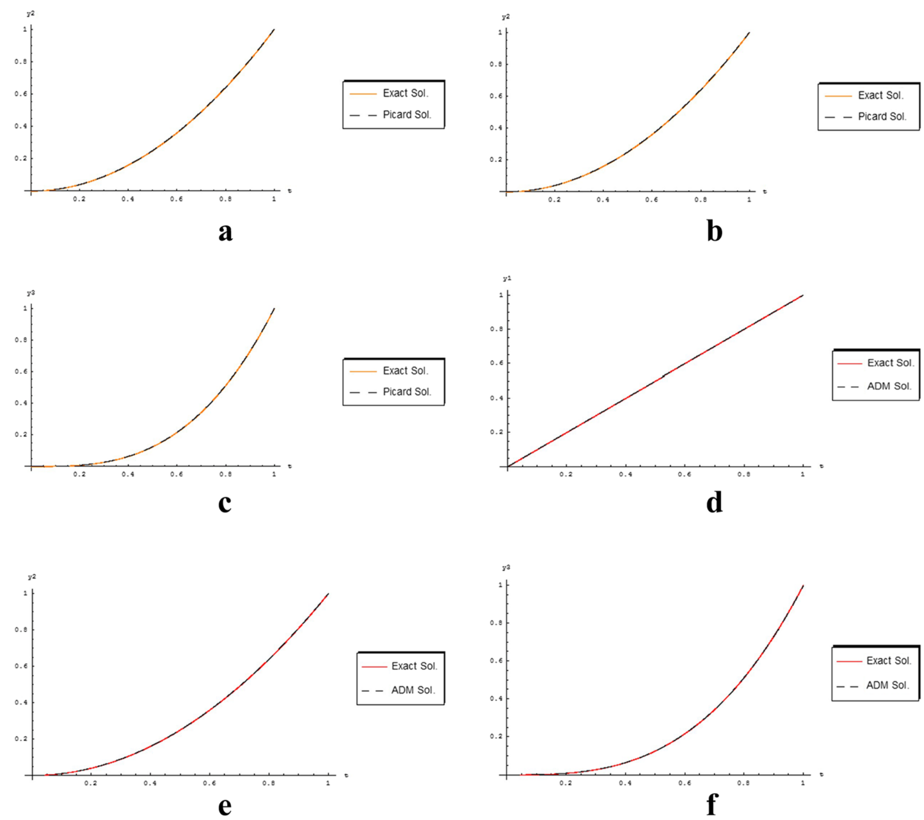

Fig. 2 **a** Picard and exact solution of y_1. **b** Picard and exact solution of y_2. **c** Picard and exact solution of y_3. **d** ADM and exact sol. of y_1. **e** ADM and exact solution of y_2. **f** ADM and exact solution of y_3

Table 1 Absolute relative error between exact, Picard and ADM solutions of y_1

t	***Absolute relative error between*** **Exact and Picard Solutions of** $\boldsymbol{y_1}$	***Absolute relative error between*** **Exact and ADM Solutions of** $\boldsymbol{y_1}$
0.1	4.06666×10^{-23}	2.06044×10^{-15}
0.2	1.06605×10^{-16}	8.43956×10^{-12}
0.3	1.5755×10^{-14}	1.095×10^{-9}
0.4	2.79452×10^{-12}	3.45681×10^{-8}
0.5	1.55117×10^{-10}	5.03017×10^{-7}
0.6	4.129×10^{-9}	4.48461×10^{-6}
0.7	6.61776×10^{-8}	0.0000285108
0.8	7.31484×10^{-7}	0.0001415
0.9	6.08514×10^{-6}	0.000581157
1	0.0000404295	0.00205527

$$\begin{aligned} {}^{AB}\mathcal{D}_t^{\alpha} y_1 &= 1 - y_1, \\ {}^{AB}\mathcal{D}_t^{\alpha} y_2 &= y_1 - y_2^2, \\ {}^{AB}\mathcal{D}_t^{\alpha} y_3 &= y_2^2, \end{aligned} \tag{20}$$

Table 2 Absolute relative error between exact, Picard and ADM solutions of y_2

t	*Absolute relative error between* **Exact and Picard Solutions of y_2**	*Absolute relative error between* **Exact and ADM Solutions of y_2**
0.1	2.3238×10^{-24}	2.47608×10^{-16}
0.2	6.09171×10^{-19}	9.79546×10^{-13}
0.3	9.00286×10^{-16}	1.27098×10^{-10}
0.4	1.59687×10^{-13}	2.01227×10^{-9}
0.5	8.86392×10^{-12}	5.83804×10^{-8}
0.6	2.3595×10^{-10}	5.0386×10^{-7}
0.7	3.78188×10^{-9}	3.30691×10^{-6}
0.8	4.18068×10^{-8}	0.0000163978
0.9	3.47854×10^{-7}	0.0000672372
1	2.31195×10^{-6}	0.000237109

Table 3 Absolute relative error between exact, Picard and ADM solutions of y_3

t	*Absolute relative error between* **Exact and Picard Solutions of y_3**	*Absolute relative error between* **Exact and ADM Solutions of y_3**
0.1	1.32789×10^{-23}	4.01786×10^{-15}
0.2	3.48098×10^{-18}	1.64571×10^{-11}
0.3	5.14449×10^{-15}	2.13521×10^{-9}
0.4	9.12499×10^{-13}	6.74004×10^{-8}
0.5	5.06509×10^{-11}	9.80474×10^{-15}
0.6	1.34828×10^{-9}	8.73407×10^{-6}
0.7	2.16106×10^{-8}	0.0000554211
0.8	2.38892×10^{-7}	0.000273993
0.9	1.98767×10^{-6}	0.00111721
1	0.0000132102	0.00390145

Subject to the initial conditions,

$$y_k(0) = 0, \quad k = 1, 2, 3.$$

Where $0 < \alpha < 1$.

1- **ADM Solution:**

Apply ${}^{AB}I^{\alpha}$ to the systems (20), then using ADM and replace each nonlinear term by its corresponding Adomian polynomials we obtain,

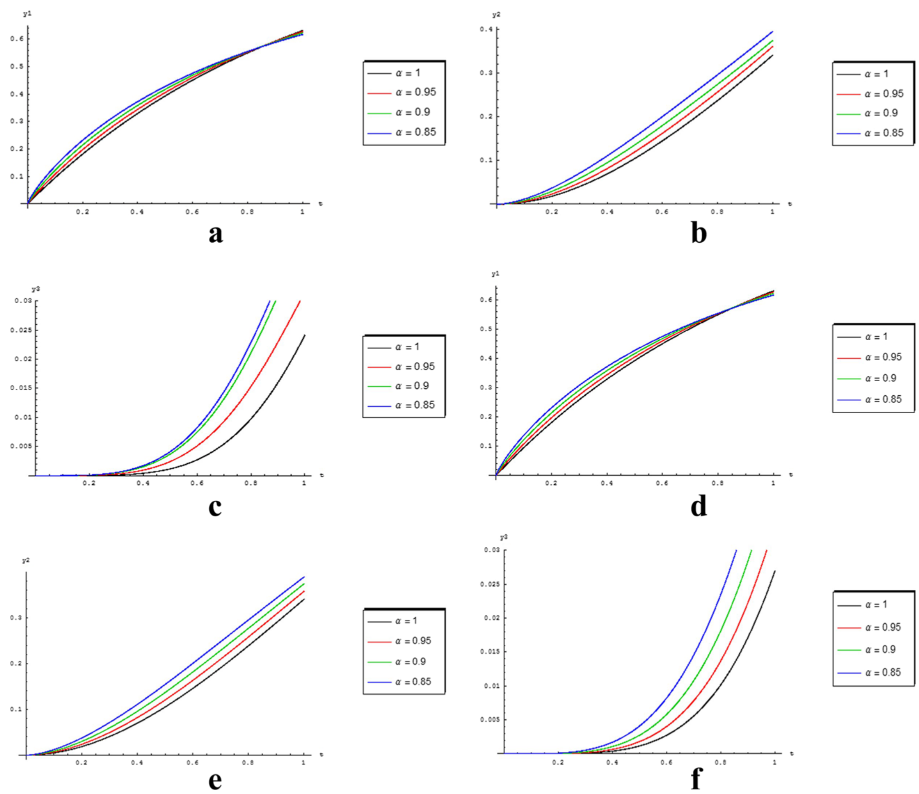

Fig. 3 **a** ADM solution of y_1. **b** ADM solution of y_2. **c** ADM solution of y_3. **d** Picard solution of y_1. **e** Picard solution of y_2. **f** Picard Solution of y_3

$$
\begin{array}{l}
y_{1,0}=\frac{t^{\alpha}}{\Gamma(1+\alpha)},\quad y_{1,j+1}=-{}^{AB}I^{\alpha}[y_{1,j}],\\
y_{2,0}=0,\quad y_{2,j+1}={}^{AB}I^{\alpha}[y_{1,j}-A_j],\\
y_{3,0}=0,\quad y_{3,j+1}={}^{AB}I^{\alpha}[A_j].
\end{array}
\tag{21}
$$

Moreover, the final solution will be,

$$y_1=\sum_{i=0}^{\infty}y_{1,n},y_2=\sum_{i=0}^{\infty}y_{2,n},y_3=\sum_{i=0}^{\infty}y_{3,n}.$$

2- **Picard Solution:**

Using Picard method to the systems (20), the solution algorithm will be,

$$
\begin{array}{l}
y_{1,0}=\frac{t^{\alpha}}{\Gamma(1+\alpha)},\quad y_{1,j+1}=y_{1,0}-{}^{AB}I^{\alpha}[y_{1,j}],\\
y_{2,0}=0,\quad y_{2,j+1}=y_{2,0}+{}^{AB}I^{\alpha}\left[y_{1,j}-(y_{2,j})^2\right],\\
y_{3,0}=0,\quad y_{3,j+1}=y_{3,0}+{}^{AB}I^{\alpha}[(y_{2,j})^2].
\end{array}
\tag{22}
$$

Moreover, the final solution will be,

$$y_1=\lim_{n\to\infty}y_{1,n},y_2=\lim_{n\to\infty}y_{2,n},y_3=\lim_{n\to\infty}y_{3,n}.$$

Figure 3a-c show ADM solutions of y_1, y_2 and y_3 at different values of α $(\alpha=1,0.95,0.9,0.85)$.

While, Fig. 3d-f show Picard solutions of y_1, y_2 and y_3 at the same values of α.

Comparing between Fig. 3a-c and d-f, we see that ADM solutions of y_1, y_2 and y_3 coincide with Picard solutions at the same values of α.

Example 4. Consider the following nonlinear system of FDEs,

$$\begin{aligned} {}^{AB}\mathcal{D}_t^{\alpha} y_1 &= y_1^2 + y_2, \\ {}^{AB}\mathcal{D}_t^{\alpha} y_2 &= 1 + y_2 \cos y_1, \end{aligned} \tag{23}$$

Subject to the initial conditions,

$$y_k(0) = 0, \quad k = 1, 2.$$

Where $\alpha \in (0, 1)$.

1- **ADM solution:**

Using ADM to system (23) leads to the following scheme,

$$\begin{aligned} y_{1,0} = 0, \quad & y_{1,j+1} = {}^{AB}I^{\alpha}(A_{1,j}) + {}^{AB}I^{\alpha}(y_{2,j}), \\ y_{2,0} = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad & y_{2,j+1} = {}^{AB}I^{\alpha}\left(A_{2,j}\right), \end{aligned} \tag{24}$$

Where $A_{1,j}$ and $A_{2,j}$ represent the Adomian polynomials of the nonlinear terms y_1^2 and $y_2 \cos y_1$ respectively.

Moreover, the final solution will be,

$$y_1 = \sum_{i=0}^{\infty} y_{1,n}, y_2 = \sum_{i=0}^{\infty} y_{2,n}$$

2- **Picard Solution:**

Using Picard method to the system (23), the solution algorithm will be,

$$\begin{aligned} y_{1,0} = 0, \quad & y_{1,j+1} = y_{1,0} + {}^{AB}I^{\alpha}[(y_{1,j})^2 + y_{2,j}], \\ y_{2,0} = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad & y_{2,j+1} = y_{2,0} + {}^{AB}I^{\alpha}\left[y_{2,j} cos(y_{1,j})\right], \end{aligned} \tag{25}$$

Moreover, the final solution will be,

$$y_1 = \lim_{n \to \infty} y_{1,n}, y_2 = \lim_{n \to \infty} y_{2,n}.$$

Figure 4a and b show ADM solutions of y_1 and y_2 at different values of α $(\alpha = 1, 0.95, 0.9, 0.85, 0.8)$.

While, Fig. 4c and d show Picard solutions of y_1 and y_2 at the same values of α.

Figure 4e and f show ADM solution of y_1 and y_2 at another different values of α $(\alpha = 0.25, \alpha = 0.5, \alpha = 0.75, \alpha = 1)$. While, Fig. 4g and h show Picard solutions of y_1 and y_2 at the same values of α.

Example 5. Consider the following nonlinear system of FDEs,

$$\begin{aligned} {}^{AB}\mathcal{D}_t^{\alpha} y_1 &= 2y_2^2, \\ {}^{AB}\mathcal{D}_t^{\alpha} y_2 &= 1 + t y_1, \\ {}^{AB}\mathcal{D}_t^{\alpha} y_3 &= 1 + y_2 y_3, \end{aligned} \tag{26}$$

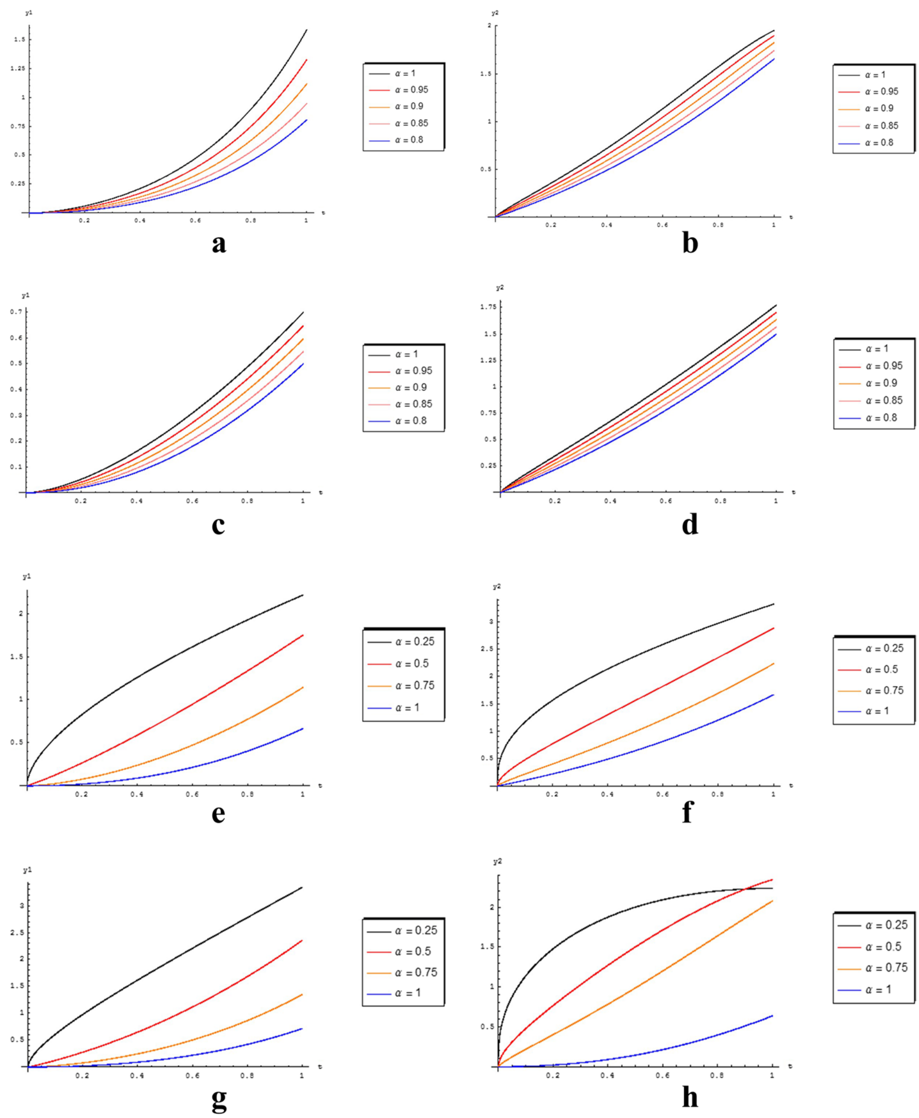

Fig. 4 **a** ADM solution of y_1 [$n=5$]. **b** ADM solution of y_2 [$n=5$]. **c** Picard solution of y_1 [$n=1$]. **d** Picard solution of y_2 [$n=1$]. **e** ADM solution of y_1 [$n=3$]. **f** ADM solution of y_2 [$n=3$]. **g** Picard solution of y_1 [$n=2$]. **h** Picard solution of y_2 [$n=2$]

Subject to the initial conditions,

$$y_k(0) = 0, \quad k = 1, 2, 3.$$

Where $\alpha \in (0, 1)$.

1- **ADM solution:**

Applying ADM to system (26) leads to the following recursive relations,

$$y_{1,0} = 0, \quad y_{1,j+1} = {}^{AB}I^{\alpha}(2A_{1,j}), \tag{27}$$

$$y_{2,0} = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad y_{2,j+1} = {}^{AB}I^{\alpha}\left(ty_{1,j}\right), \tag{28}$$

$$y_{3,0} = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad y_{3,j+1} = {}^{AB}I^{\alpha}\left(A_{2,j}\right), \tag{29}$$

Where $A_{1,j}$ and $A_{2,j}$ represent the Adomian polynomials of the nonlinear terms y_2^2 and y_2y_3 respectively.

Moreover, the final solution will be,

$$y_1 = \sum_{i=0}^{\infty} y_{1,n}, y_2 = \sum_{i=0}^{\infty} y_{2,n}, y_3 = \sum_{i=0}^{\infty} y_{3,n}.$$

2- **Picard Solution:**

Using Picard method to the system (26), the solution algorithm will be,

$$\begin{aligned} y_{1,0} &= 0, \quad y_{1,j+1} = y_{1,0} + {}^{AB}I^{\alpha}[(y_{2,j})^2], \\ y_{2,0} &= \tfrac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad y_{2,j+1} = y_{2,0} + {}^{AB}I^{\alpha}\left[ty_{1,j}\right], \\ y_{3,0} &= \tfrac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad y_{3,j+1} = y_{3,0} + {}^{AB}I^{\alpha}\left[y_{1,j}y_{2,j}\right], \end{aligned} \tag{30}$$

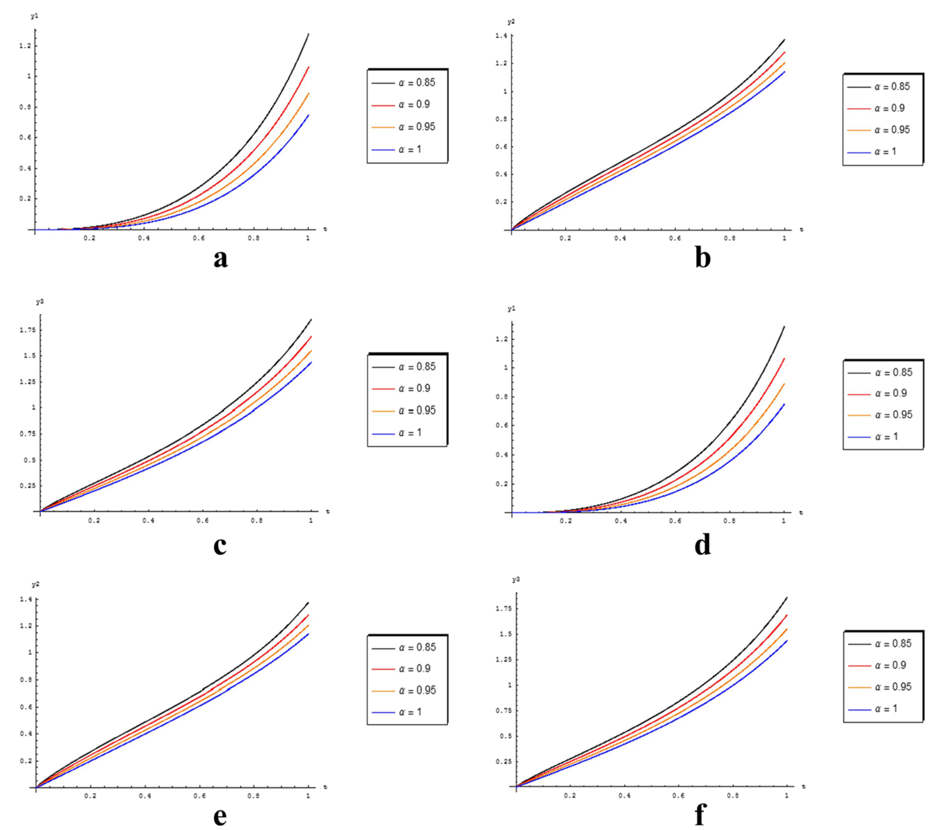

Fig. 5 **a** ADM solution of y_1 [$n=5$]. **b** ADM solution of y_2 [$n=5$]. **c** ADM solution of y_3 [$n=5$]. **d** Picard solution of y_1 [$n=5$]. **e** Picard solution of y_2 [$n=5$]. **f** Picard solution of y_3 [$n=5$]

Moreover, the final solution will be,

$$y_1 = \lim_{n\to\infty} y_{1,n}, y_2 = \lim_{n\to\infty} y_{2,n}, y_3 = \lim_{n\to\infty} y_{3,n}.$$

Figure 4a-c show ADM solutions of y_1, y_2 and y_3 at different values of α ($\alpha = 0.85, 0.9, 0.95, 1$).

While, Fig. 4d-f show Picard solutions of y_1, y_2 and y_3 at the same values of α.

We see from the above figures that ADM solutions of (y_1, y_2 *and* y_3) are coincide with Picard solutions at the same values of α.

Conclusions

In this research, we use two interesting methods (ADM and Picard methods) to solve a system of nonlinear fractional differential equations of Atangana–Baleanu sense; these two methods give analytical solutions, which coincide with each other (see Figs. 1, 2, 3, 4 and 5). In addition, these two methods give good approximate analytical solutions as we compared them with the exact solution (see Example 2) and from these results, we see that Picard method give more accurate results than ADM but ADM take less time than Picard (see Tables 1, 2 and 3).

Abbreviations

ADM Adomian Decomposition Method
FDEs Fractional differential equations

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Author's contributions

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